Bose-Einstein condensation in harmonic double wells

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Abstract

We discuss Bose-Einstein condensation in harmonic traps where the confinement has undergone a splitting along one direction. We mostly consider the 3D potentials consisting of two cylindrical wells separated a distance $2a$ along the z-axis. For ideal gases, the thermodynamics of the confined bosons has been investigated performing exact numerical summations to describe the major details of the transition and comparing the results with the semiclassical density-of-states approximation. We find that for large particle number and increasing well separation, the condensation temperature evolves from the thermodynamic limit value $\mathrm{T}_c^{(0)}(N)$ to $\mathrm{T}_c^{(0)}(N/2)$. The effects of adding a repulsive interaction between atoms has been examined resorting to the Gross-Pitaevskii-Popov procedure and it is found that the shift of the condensation temperature exhibits different signs according to the separation between wells. In particular, for sufficiently large splitting, the trend opposes the well known results for harmonic traps, since the critical temperature appears to increase with growing repulsion strength.

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I. INTRODUCTION

The recent observation of Bose-Einstein condensation (BEC) of atomic alkali gases subjected to magnetic or magnetooptic traps [1–3] has triggered an important amount of related work. In fact, since the possibility of experimental realization of BEC was put forward, sizeable theoretical effort addressed both the quantum and the thermodynamical aspects of this phenomenon when confined boson systems are involved. Starting from the earliest approaches to this field [4–7], extensive work has been devoted to the various properties of harmonic traps [8–10,12–23], which can account for most of the characteristic scales of experimentally achieved BEC according to very intuitive scaling arguments based on energy balance [8]. The main features of BEC of ideal gases in either isotropic or anisotropic quadratic potential wells have been analyzed by various authors [9,10,12–19], paying special attention to the modifications of the transition patterns due to the finite number of trapped atoms.

The nature of the transition itself has been the subject of some controversy. Kirsten and Toms have claimed that no spontaneous symmetry breaking leading to BEC can happen in confined potentials [9,10], as evidenced by the fact that the chemical potential of the bosons does not reach the value of the ground state energy at a finite temperature. However, this behavior of the chemical potential does not prevent the existence of a noticeable peak in the specific heat of the gas at a well defined temperature, for dimensions other than unity, coinciding with a sudden important rise in the ground state occupation number. This characteristic has been recognized by various authors [10–12] as the signature of a phase transition. Furthermore, calculation methods based on exact summations [12–15] and Euler-MacLaurin approximation formulae [16,17] yield results practically identical to those arising from more sophisticated treatments [9,10,18]. Bose condensed gases in anisotropic traps have also deserved attention, especially in view of the fact that experimental devices lead to the possibility that the definite stages of condensation are governed by the most binding 1D forces [17,19]. In particular, it has been clearly established that in highly anisotropic

potentials, the peak in the specific heat is linked to the freezing of those degrees of freedom lying higher in the energy spectrum, and macroscopic occupation of the ground state of the system occurs at a lower temperature [19].

The effects of two particle interactions represented by a single parameter, the scattering length, have been examined [6,20–23]; the solutions of the Gross-Pitaevskii (GP) equation provide information upon the condensate wave function, and it has been shown that the transition temperature is sensitive to the sign of the interparticle forces in a way that opposes the expected results for the free homogeneous gas [22]. The variational solution to the GP equation [23] also provides an interesting means to approach the various features of BEC of weakly interacting gases.

More recently, double Bose condensates have become an attractive field of research in view of the close resemblance between these systems and other bistable devices familiar in quantum optics [24]. The properties of double traps have been mostly investigated from the perspective of these analogies, keeping in mind that due to the large separations between the experimentally split traps and the significant height of the halving barriers, the condensate wave function essentially corresponds to two nonoverlapping wells. It is our present purpose to perform a detailed study of the quantum and thermodynamic aspects of these bistable traps in terms of the shape of the barrier and the well separation in the framework of a simple description. For this sake, in Sec. 2 we present the essential formulae for the quantum mechanics and the thermodynamics of an ideal gas in a harmonic double well, while the results of the calculation are presented in Sec. 3. Section 4 is devoted to the weakly interacting gas described by means of the GP + Popov formalism. The conclusions are summarized in Sec. 4.

II. THE HARMONIC DOUBLE WELL

The spectrum of the 1D harmonic double well with potential

$$V(z) = \frac{m\,\omega^2}{2}\,(|z| - a)^2 \tag{2.1}$$

can be represented [25] by the parabolic cylinder functions $D_\nu[\sqrt{2m\,\omega\,/\hbar}\,(|z|-a)]$ with eigenvalues $\varepsilon_\nu = \hbar\,\omega(\nu+1/2)$. The quantum numbers ν are determined establishing that the eigenfunctions be either even or odd and it is found that the spectrum evolves from a purely harmonic one at separation $a=0$ to a doubly degenerate harmonic one of the same frequency , as a grows indefinitely. Furthermore, we shall consider 2D (*i.e.*, $\omega_y = 0$) and 3D traps of the form

$$V(\mathbf{r}) = \frac{m}{2}\,\left[\omega_x^2\,x^2 + \omega_y^2\,y^2 + \omega_z^2\,(|z|-a)^2\right] \tag{2.2}$$

A numerical computation of the energy levels of (2.1) enables us to examine the thermodynamic properties of a confined boson gas by straightforward summation of the relation defining the internal energy,

$$U = \sum_i N_i\,\varepsilon_i \tag{2.3}$$

where $N_i = \left[e^{\beta(\varepsilon_i-\mu)} - 1\right]^{-1}$ is the boson occupation number at temperature $T = 1/\beta$ for the level with energy $\varepsilon_i = \hbar\,(n_x\,\omega_x + n_y\,\omega_y + \nu\,\omega_z + 3/2)$, once the particle number equation

$$N = \sum_i n_i \tag{2.4}$$

has been solved to determine the chemical potential μ for the given value of N. The specific heat is afterwards computed differentiating the energy with respect to temperature.

In this work we compare the results of the exact numerical calculation with those of the semiclassical method derived by Bagnato *et al* [6], which in the present case consists of replacing the summation in (2.4) by

$$N = N_0 + N_1 + \sum_j z^j \int d\varepsilon\,\rho(\varepsilon)\,e^{-\dfrac{j\,\varepsilon}{kT}} \tag{2.5}$$

where $\rho(\varepsilon)$ is the classical extension of the density of states for single particles in the well. Notice that in Eq. (2.5) we explicitely separate the populations N_0 of the ground state and N_1 of the first excited state, since as the height of the barrier $V_0 = m\,\omega^2\,a^2/2$ increases, these two energy levels approach the doubly degenerate ground state of the infinitely distant wells.

The density of states is computed according to the prescription given in Ref. [6], which for an arbitrary confining D-dimensional potential $V(\mathbf{r})$ can be cast into the most general form

$$\rho_D(\varepsilon) = \frac{C_D}{h^D}\, 2^{D/2-1}\, m^{D/2} \int d^D r\, [\varepsilon - V(\mathbf{r})]^{D/2-1} \tag{2.6}$$

where the integral is to be carried within the classically allowed region. In view of the relation

$$\rho_D(\varepsilon) = \int_0^\varepsilon d\varepsilon'\, \rho_{D-1}(\varepsilon - \varepsilon')\, \rho_1(\varepsilon') \tag{2.7}$$

we only need to compute the 1D density for the double well. The 3D case is studied convoluting ρ_1 with

$$\rho_2(\varepsilon) = \frac{1}{\hbar\,\omega} \left(\frac{\varepsilon}{\hbar\,\omega} + 1 \right) \tag{2.8}$$

for $\omega = \omega_x = \omega_y$, which yields a formula for the number of particles in two dimensions that coincides with the high temperature limit of the exact summation (see Appendix). It should be noticed that the semiclassical prescription of Ref. [6] only gives the first term of Eq. (2.8) for the 2D density.

III. THE SYMMETRIC DOUBLE WELL

We have investigated BEC of an ideal boson gas confined in the trap described in the preceding section for dimensions D=1 to 3, comparing the results of the exact summations with those of the approximate density-of-states method, in terms of the well splitting a and the number of particles N. In the 3D case, we also examine the anisotropic potentials mostly related to experimental situations; notice that in spite of the symmetry breaking along the z-direction, we shall consider the potential to be isotropic when $\omega_x = \omega_y = \omega_z$ and anisotropic otherwise. The analytical expressions for the semiclassical densities of states computed according to the prescription (2.6) and the corresponding numbers of particles are summarized in the Appendix.

The specific calculations indicate that the 1D well exhibits the characteristic sharp increase of the occupation number N_0 around a temperature T$_c$, however this is the usual seudotransition that is not accompanied by a peak in the specific heat, as shown in Fig. 1 for various separations between harmonic wells (hereafter indicated by the dimensionless variable $z_0 = \sqrt{2m\,\omega/\hbar\,a})$. We can appreciate the moderate sensitivity of this quantity to the splitting z_0. We also compare the exact ground state $\psi(z)$ of the double well with the wave function

$$\tilde{\psi}(z) = \mathcal{N}\ [\psi_0(z-a) + \psi_0(z+a)] \tag{3.1}$$

where $\psi_0(z \pm a)$ represents the ground state of a single harmonic well of the same frequency centered at $\pm a$ and $\mathcal{N}$ is a normalization factor. This is illustrated in Fig. 2, where we plot, for z_0 =1 and 3, (left and right columns, respectively) the functions $\psi(z)$ and $\tilde{\psi}(z)$ in full and dashed lines, respectively. In the upper plots, each single ground state wave function $\psi_0(z \pm a)$ has been individually normalized, whereas in the lower ones the sum (3.1) has been normalized to unity. The dotted lines indicate each separate contribution to $\tilde{\psi}(z)$. It is clear that for every well separation here considered, the exact wave function resembles almost exactly the normalized combination (3.1) shown in the lower pictures.

Let us now consider a 3D isotropic trap; no substantial qualitative differences have been observed in the 2D case. In Fig. 3 we exhibit the occupation numbers N_0 and N_1 as functions of the temperature in units of the critical temperature $T_c^{(0)}$ corresponding to infinite number of atoms, for well separations $z_0 = 0$ (*i.e.*, a single harmonic well) and 1, as well as for total particle numbers N=100 and 10000. We observe a rather sharp transition at a temperature slightly lower than unity for sufficiently large number of particles; furthermore, this transition temperature decreases with growing barrier height. On the other hand, the larger the value of z_0, the closer the resemblance between N_0 and N_1 even for temperatures substantially below $T_c^{(0)}$; this fact reflects the quasidegeneracy of the ground state and the first excited state for large finite values of z_0, which evolves into complete degeneracy at infinite separation. This can be visualized in Fig. 4, where we have selected N=1000 and various barrier heights

V_0, which allow us to appreciate the significant decrease of the transition temperature that occurs as the wells split apart.

The existence of a transition becomes evident as one analyzes the specific heat. Figs. 5 and 6 correspond to the same parameters as Figs. 3 and 4, respectively; while lines indicate numerical results, symbols locate the calculations performed with the semiclassical expressions listed in the Appendix. We observe that the agreement between exact and approximate results is almost perfect at the lowest temperatures, whereas for high temperatures, that agreement holds only for large particle numbers. For such numbers the semiclassical approach locates the transition to an acceptable accuracy, except for some sligth overestimation of both the characteristic temperature and the height of the peak; however, this departure is very noticeable for small quantities of atoms. In Fig. 6 we may notice the consequences of increasing the well separation: the major effect takes place for a splitting $z_0 = 3$, with only minor differences as one enlarges this number.

We have also performed calculations for anisotropic traps; as an illustration, in Fig. 7 we show the specific heat for 1000 atoms and well separations $z_0 = 0$ and 1. The various curves correspond to the aspect ratios of the JILA [1] and MIT traps [2], namely $\omega_z/\omega_{x,y} = \sqrt{8}$ and ($\omega_z/\omega_x = 3.2$, $\omega_z/\omega_y = 1.8$), respectively, and to an isotropic well with a frequency $\omega = 0.56\,\omega_z$ equal to the MIT geometric mean. We realize that the larger the anisotropy, the smaller is the transition temperature and the lower is the height of the peak. This also happens when one switches from the single to the double well, however to a less significant extent as compared to the anisotropy effect.

IV. THE WEAKLY INTERACTING GAS IN THE DOUBLE WELL

The influence of weak to moderate interactions between trapped particles can be investigated in a GP mean field approach generalized for finite temperatures. Within this formalism, the condensate wavefunction $\psi(r)$ is obtained from the nonlinear equation [27]

$$\hat{h}_0\,\psi(r) = \mu\,\psi(r) \tag{4.1}$$

with

$$\hat{h}_0 = \left\{ -\frac{\nabla^2}{2\,m} + V(r) + g\left[n_c(r) + 2\,\tilde{n}(r)\right] \right\} \tag{4.2}$$

where we have made the usual decomposition of the density into condensate and noncondensate contributions, i.e., $n(r) = n_c(r) + \tilde{n}(r)$, and $n_c(r) = |\psi(r)|^2$. The terms involving the interaction strength g arise from a two-body seudopotential $g\,\delta(\mathbf{r})$. In the s-wave approximation, which is adequate for very dilute gases, one has $g = 4\pi\,\alpha\,\hbar^2/m$, being α the scattering length.

Within the Popov approximation of the Hartree-Fock-Bogoliubov (HFB) theory, the excitations of a dilute system of bosons at low and intermediate temperatures are given by the coupled eigenvalue equations

$$\begin{aligned} \hat{\mathcal{L}}\,u_i(r) - g\,n_c(r)\,v_i(r) &= E_i\,u_i(r) \\ \hat{\mathcal{L}}\,v_i(r) - g\,n_c(r)\,u_i(r) &= -E_i\,v_i(r) \end{aligned} \tag{4.3}$$

which define the quasiparticle energies E_i and amplitudes u_i and v_i, with

$$\hat{\mathcal{L}} \equiv \hat{h}_0 + g\,n_c(r) \tag{4.4}$$

The noncondensate density is related to the quasiparticle properties according to [27,28]

$$\tilde{n}(r) = \sum_i \left\{ |v_i(\mathbf{r})|^2 + \left[|u_i(\mathbf{r})|^2 + |v_i(\mathbf{r})|^2 \right] N(E_i) \right\} \tag{4.5}$$

where $N(E_i) = \left(e^{\beta E_i} - 1\right)^{-1}$ is the Bose-Einstein distribution of the quasiparticles.

One must solve the coupled GP (4.1) and HFB (4.3) equations, using the self-consistent densities $n_c(r)$ and $\tilde{n}(r)$ for fixed number of particles N. For this sake, we use the decoupling procedure introduced by Hutchinson *et al* [29], and expand $\tilde{h}_0$ in a basis of eigenfunctions of the 2D harmonic oscillator on the (x, y) plane times a discretized basis set for the 1D double well along the z-axis. To illustrate this procedure, in Fig. 8 we plot the condensate fraction as a function of the temperature, for 100 particles at various barrier heights and positive interaction strengths $s = \alpha\sqrt{m\,\omega_z/\hbar}$. We observe that in all cases, the system

reaches BEC at temperatures below the transition temperature for a single well with no interactions. However, depending on the value of z_0 two different behaviors occur. When z_0 is small, the condensation temperature decreases, the stronger the interaction, in agreement with the well known result for the harmonic trap [6,22]. Conversely, if z_0 is high enough, an increase of s yields a higher critical temperature. This feature can be interpreted if we consider the single particle spectrum in the mean field, whose ground and first excited states are depicted in Fig. 9 for a temperature T=0.26 $T_c^{(0)}$ and N=100. The overall trend is similar at any temperature and this choice is a convenient one in view of the displayed scale. We realize that these levels also exhibit two tendencies: for small well separations, as the interaction is enlarged the energy levels get closer, giving a more degenerate spectrum and a higher density of states, which in turn demand further cooling in order to achieve BEC. On the other hand, if z_0 is sufficiently large (above roughly z_0=2.5 in Fig. 9) the ground and first excited states of a double well are already quasi-degenerate; in this case, the effect of the interaction as indicated in this figure is to uniformly raise the whole spectrum. As a consequence, due to level crossing, particles occupying the first excited state of the free gas can be promoted to the interacting ground state at no thermodynamic cost, giving thus rise to a moderate increase of the condensation temperature.

V. DISCUSSION AND SUMMARY

In this work we have discussed the characteristics of Bose-Einstein condensation in harmonic wells where the confinement has undergone some splitting along one of the three directions. We mostly consider the 3D potentials consisting of two cylindrical wells separated a distance $2a$ along the z-axis. For ideal gases, the thermodynamics of the confined bosons has been investigated performing exact numerical summations for the particle number, which enables us to determine the chemical potential, and for the total energy, after which we derive the specific heat by means of a numerical differentiation with respect to temperature. This procedure allows us to describe the major details of the transition and

to compare the results with the approximation corresponding to replacement of exact summations by integrals weighted by a semiclassical density of states.

The general results can be summarized as follows. We find that the semiclassical approach is capable of locating the transition, *i.e.*, the peak in the specific heat, with higher accuracy the larger the amount of trapped atoms. The overestimation in both the precise value of the transition temperature and the height of the peak is rather pronounced for small systems and becomes progressively less important as the system approaches the thermodynamic limit. It becomes clear that the effect of halving the population is to lower the transition temperature; for large particle number, the trend of this temperature as the well separation grows from zero to infinity is clearly to sweep the path between critical temperatures $T_c(N)$ and $T_c(N/2)$. This general behavior is independent of the dimensionality of the system; however, just as in the single well problem [12,14], no strict phase transition takes place in the 1D potential. In fact, in this case one finds that in spite of the important rise in the slope of the occupation number N_0 observed at a rather well defined temperature, the specific heat increases monotonically towards the classical limit.

The effects of adding a repulsive interaction between atoms has been examined resorting to the GP + Popov procedure which yields the condensate and noncondensate densities together with the single particle spectrum. It is found that the shift of the condensation temperature exhibits different signs according to the size of the separation between wells on the z axis; in particular, for sufficiently large splitting, the trend opposes the well known results for harmonic traps, since the critical temperature appears to increase with growing repulsion strength. This unexpected feature can be explained examining the spectrum of the atoms in the mean field as a function of the well separation. One can realize that in the interacting system, not only the position of the energy levels, but the size of the gap between the ground and the first excited state, change substantially when the wells split apart. The level crossing that occurs for large separations causes atoms in the excited state to move into the new ground state, with a consequent increase in the condensation temperature.

An experimental realization of the double well is the MIT trap [2] characterized by a

large separation between wells and by a tunable barrier height. One might then wonder to what an extent the results here presented might be sensitive to independent choices of the location $\pm a$ of the minima and the barrier magnitude V_0, and intend to develop more detailed models for the specific MIT double trap. Among the simplest choices, one could quote i) a combination of two symmetric harmonic wells smoothly joined at $z = \pm a_m$ by a quadratic barrier and ii) the double well of the previous sections deformed by a central constant cutoff with value V_0 for $|z| \leq a_m$. We have performed calculations similar to those presented in section 3 for the above potentials, employing standard magnitudes of the original MIT trap [2]. We have found that the thermodynamics of the ideal gas remains identical to the previously analized double well case, in other words, the transition temperature and the overall shape of the occupation numbers and specific heat (Figs. 3 to 6) are not sensitive to the decoupling between the location of the minima and the barrier size, within the range of values of interest.

To summarize, we remark that the statistical mechanics of a bistable system can be examined on identical grounds as the case of a single equilibrium state, with an additional parameter, namely the separation between wells, as the agent of the evolution from a given number of trapped atoms up to the point where one exactly halves the population in the infinite separation limit.

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APPENDIX

We collect here various useful formulae concerning the thermodynamics of ideal gases in harmonic traps. We display here the semiclassical results obtained from the density-of-states method of Ref. [6] (Eq. (2.5) for the isotropic D-dimensional oscillator,

Particle number:

$$N = N_0 + \left(\frac{k\,T}{\hbar\,\omega}\right)^D\, g_D(z) \tag{5.1}$$

Internal energy:

$$U = D\,\hbar\,\omega\,\left(\frac{kT}{\hbar\,\omega}\right)^{D+1}\, g_{D+1}(z) \tag{5.2}$$

Specific heat:

$$\frac{C}{k} = D\,\left(\frac{k\,T}{\hbar\,\omega}\right)^D\,\left[(D+1)\,g_{D+1}(z) - D\frac{g_D^2(z)}{g_{D-1}}(z)\right] \tag{5.3}$$

whereas the exact summations give, in the high temperature limit $\hbar\,\omega << k\,T$

$$N = N_0 + \left(\frac{k\,T}{\hbar\,\omega}\right)^D\,\sum_{n=1}^{D}(-)^{n+1}\,g_n\left[z\,e^{-(n-D/2)\,\hbar\,\omega/k\,T}\right] \tag{5.4}$$

In this limit one can take advantage of the property of the Bose special functions $g_i(z)$

$$g_i\left(e^{\dfrac{\mu+\delta\mu}{k\,T}}\right) \approx g_i(z) + \frac{\delta\,\mu}{k\,T}\,\frac{g_{i-1}(z)}{z} \tag{5.5}$$

for $D \geq 2$ and write

$$N = N_0 + \left(\frac{k\,T}{\hbar\,\omega}\right)^D\,\left[g_D(z) + \frac{D}{2}\,\frac{k\,T}{\hbar\,\omega}\,\frac{g_{D-1}(z)}{z}\right] \tag{5.6}$$

from where the transition temperature is obtained specifying $N_0 = 0$ together with $z = 1$.

For the isotropic potential with the double well on the z direction, we have the following cases

Dimension D = 1:

The density of states takes the form

$$\rho_{1D}(\varepsilon) = \begin{cases} \dfrac{2}{\hbar\omega}, & \varepsilon < V_o \\ \\ \dfrac{1}{\hbar\omega} + \dfrac{2}{\pi}\,\dfrac{1}{\hbar\omega}\,arcsin\left(\sqrt{\dfrac{V_o}{\varepsilon}}\right), & \varepsilon \geq V_o \end{cases} \tag{5.7}$$

and the number of particles is

$$
\begin{aligned}
N &= N_0 + N_1 + \int_{\varepsilon_1^+}^{\infty} d\varepsilon \, \frac{\rho_{1D}(\varepsilon)}{e^{\beta(\varepsilon-\mu)} - 1} \\
&= \frac{z}{1-z} + \frac{z_1}{1-z_1} + \frac{1}{\beta\hbar\omega} \, \ln \frac{1-z_m}{(1-z_1)^2} \\
&\quad + \frac{2}{\pi} \, \frac{1}{\hbar\omega} \int_{\varepsilon_m}^{\infty} d\varepsilon \, arcsin \sqrt{\frac{V_o}{\varepsilon}} \, N(\varepsilon)
\end{aligned}
\tag{5.8}
$$

with $z = \exp(\beta\mu), z_1 = \exp\{\beta(\mu - \varepsilon_1)\}, z_m = \exp\{\beta(\mu - \varepsilon_m)\}$ and $\varepsilon_m = \max\{\varepsilon_1, V_o\}$ The total energy can be expressed, similarly to (5.8), as a summation including an integral.

Dimension D=2:

We obtain

$$
\rho_{2D}(\varepsilon) = \begin{cases} 2 \, \frac{\varepsilon}{(\hbar\omega)^2}, & \varepsilon < V_o \\ \\ \frac{1}{(\hbar\omega)^2} \left[\varepsilon + \varepsilon \, \frac{2}{\pi} \, arcsin(\sqrt{V_o/\varepsilon}) + \frac{4}{\pi} \sqrt{V_o} \, \sqrt{\varepsilon - V_o} \right], & \varepsilon \geq V_o \end{cases}
\tag{5.9}
$$

and

$$
\begin{aligned}
N &= \frac{z}{1-z} + \frac{z_1}{1-z_1} + \frac{1}{(\beta\hbar\omega)^2} \, \left[2 \, g_2(z_1) - g_2(z_m) \right] \\
&\quad + \frac{2}{\pi} \, \frac{1}{\beta(\hbar\omega)^2} \int_{\varepsilon_m}^{\infty} d\varepsilon \, arcsin \left(\sqrt{\frac{V_o}{e}} \right) \, \ln\left[1 - \exp\left(-\beta(e - \mu) \right) \right]
\end{aligned}
\tag{5.10}
$$

where $g_n(z)$ is the usual Bose function [26]

Dimension D=3:

The corresponding expressions are

$$
\rho_{3D}(\varepsilon) = \begin{cases} \frac{\varepsilon^2}{(\hbar\omega)^3}, & \varepsilon < V_o \\ \\ \frac{1}{2(\hbar\omega)^3} \left[\varepsilon^2 + \frac{2}{\pi} \, \varepsilon^2 \, arcsin \left(\sqrt{\frac{V_o}{\varepsilon}} \right) \right. \\ \\ \left. + \frac{2}{3} \, \frac{\sqrt{V_o}}{\pi} \, \sqrt{\varepsilon - V_o} \, (5\varepsilon - 2V_o) \right], & \varepsilon \geq V_o \end{cases}
\tag{5.11}
$$

and

$$
\begin{aligned}
N = & \frac{z}{1-z} + \frac{z_1}{1-z_1} + \frac{1}{(\beta\hbar\omega)^3}\ \{2\ g_3(z_1) - g_3(z_m) + \beta\hbar\omega\,[2\ g_2(z_1) - g_2(z_m)]\} \\
& + \frac{2}{\pi}\,\frac{1}{\beta^2(\hbar\omega)^3}\ \int_{\varepsilon_m}^{\infty} d\varepsilon\, arcsin\left(\sqrt{\frac{V_o}{\varepsilon}}\right)\ [g_2(z_\varepsilon) + \beta\hbar\omega\ g_1(z_\varepsilon)]
\end{aligned} \tag{5.12}
$$

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FIGURE CAPTIONS

Figure 1.- The specific heat of a 1D boson gas trapped in a double well (in units of the Boltzmann constant k_B) as a function of the reduced temperature $T/T_c^{(0)}$ for different separation between potential minima.

Figure 2.- The exact ground state probability $|\psi(z)|^2$ (full lines) of particles in the double trap as a function of the dimensionless coordinate z/a, for well separations z_0=1 (left column) and z_0=3 (right column). Dashed lines correspond to the sum of two normalized single well wave functions (upper plots) and the normalized sum (lower plots),

while dotted lines indicate the individual ground states.

Figure 3.- Occupation numbers of the ground and first excited state for total particle number N=100 (thin lines) and 10000 (thick lines) as functions of $T/T_c^{(0)}$ for the single harmonic well and for a separation equal to unity.

Figure 4.- Occupation numbers of the ground and first excited state for 1000 atoms and various barrier heights.

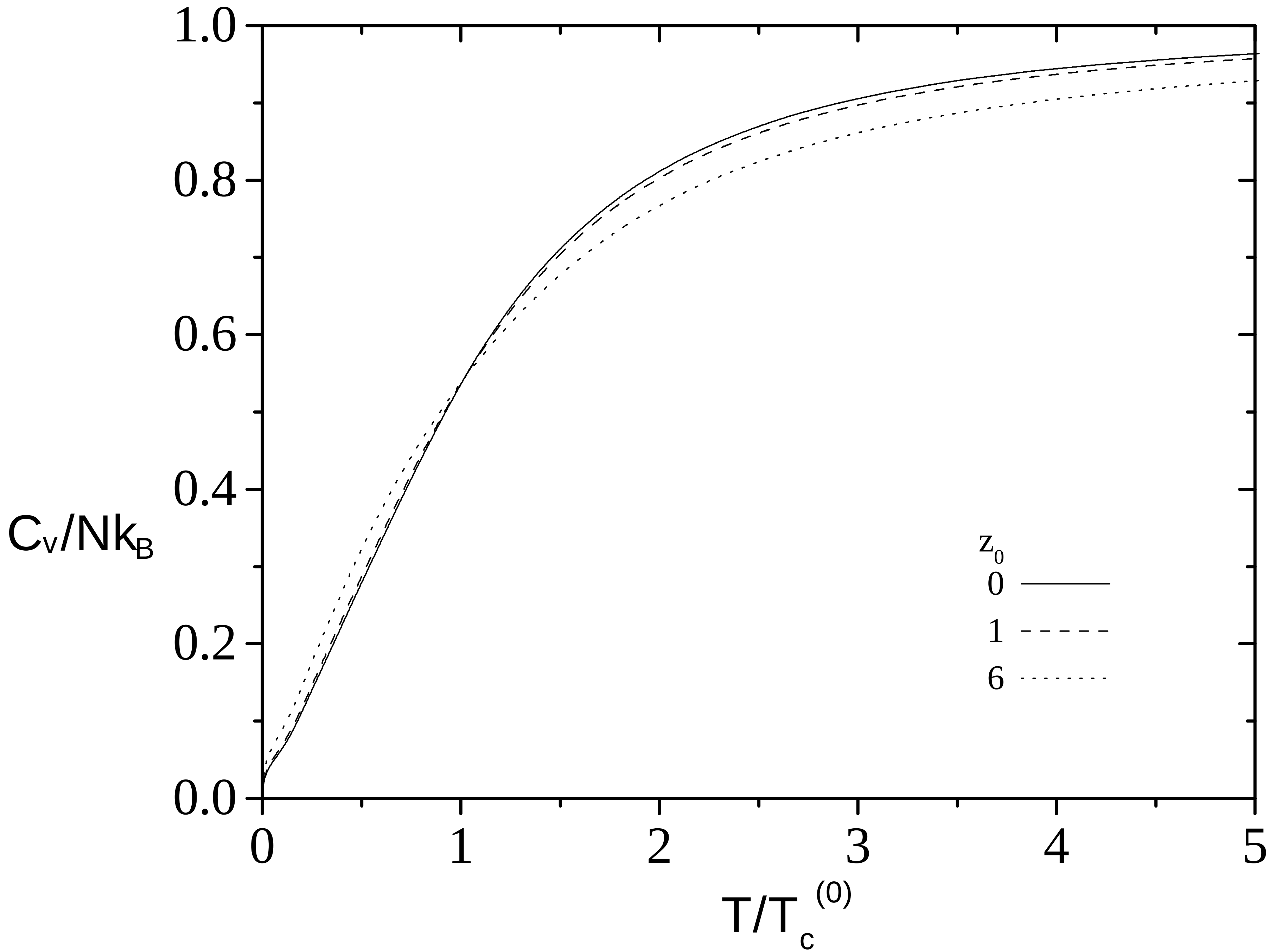
Figure 5.- Specific heat for the same situation depicted in Fig. 3. Circles and dots indicate the results of the semiclassical approximation.

Figure 6.- Same as Fig. 5 for the conditions of Fig. 4.

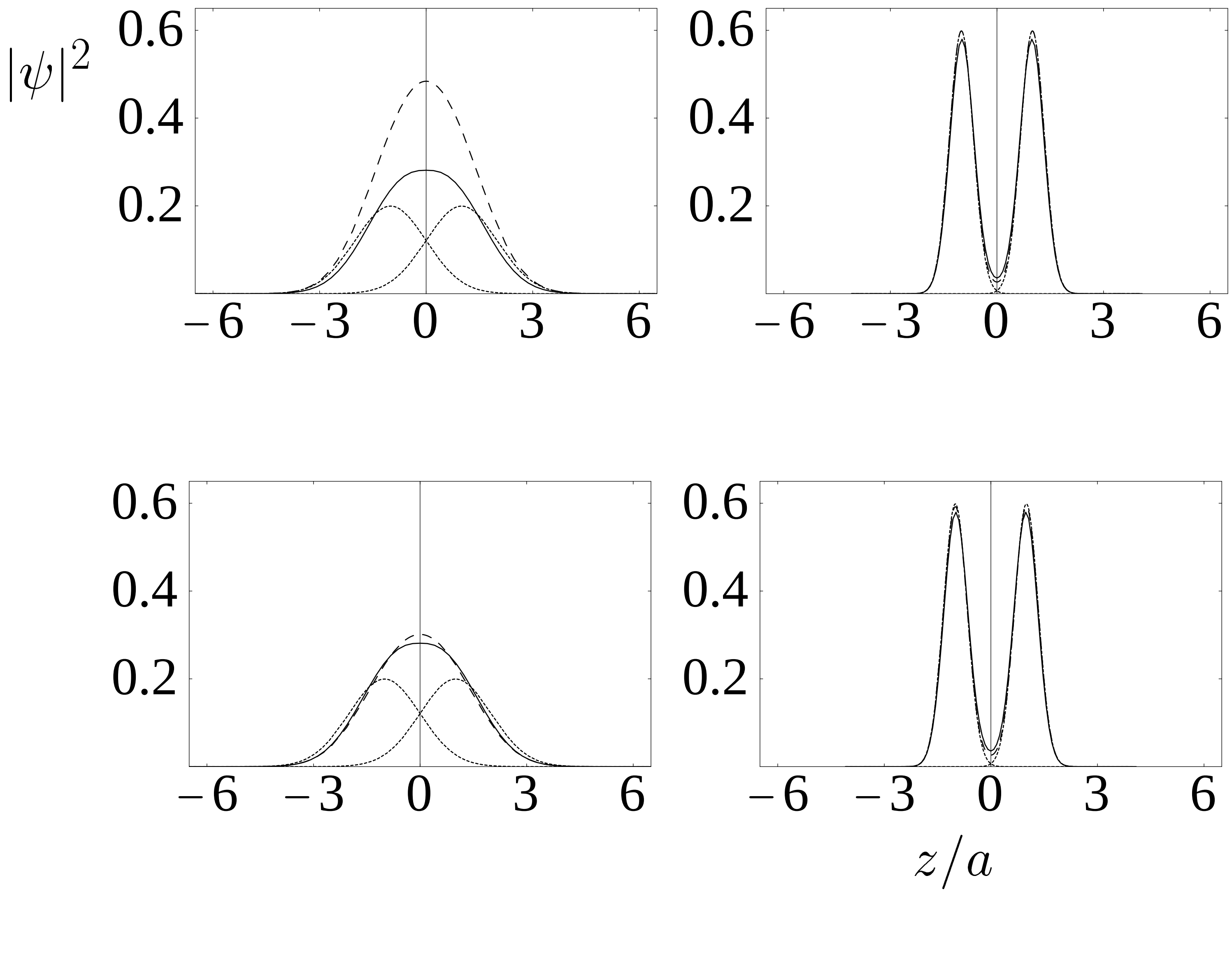
Figure 7.- Specific heat of anisotropic traps for the aspect ratios of the JILA and MIT traps, compared to an isotropic situation.

Figure 8.- Ground state occupation number of a 3D double well including weak two-body interactions, computed in the GP+Popov approximation.

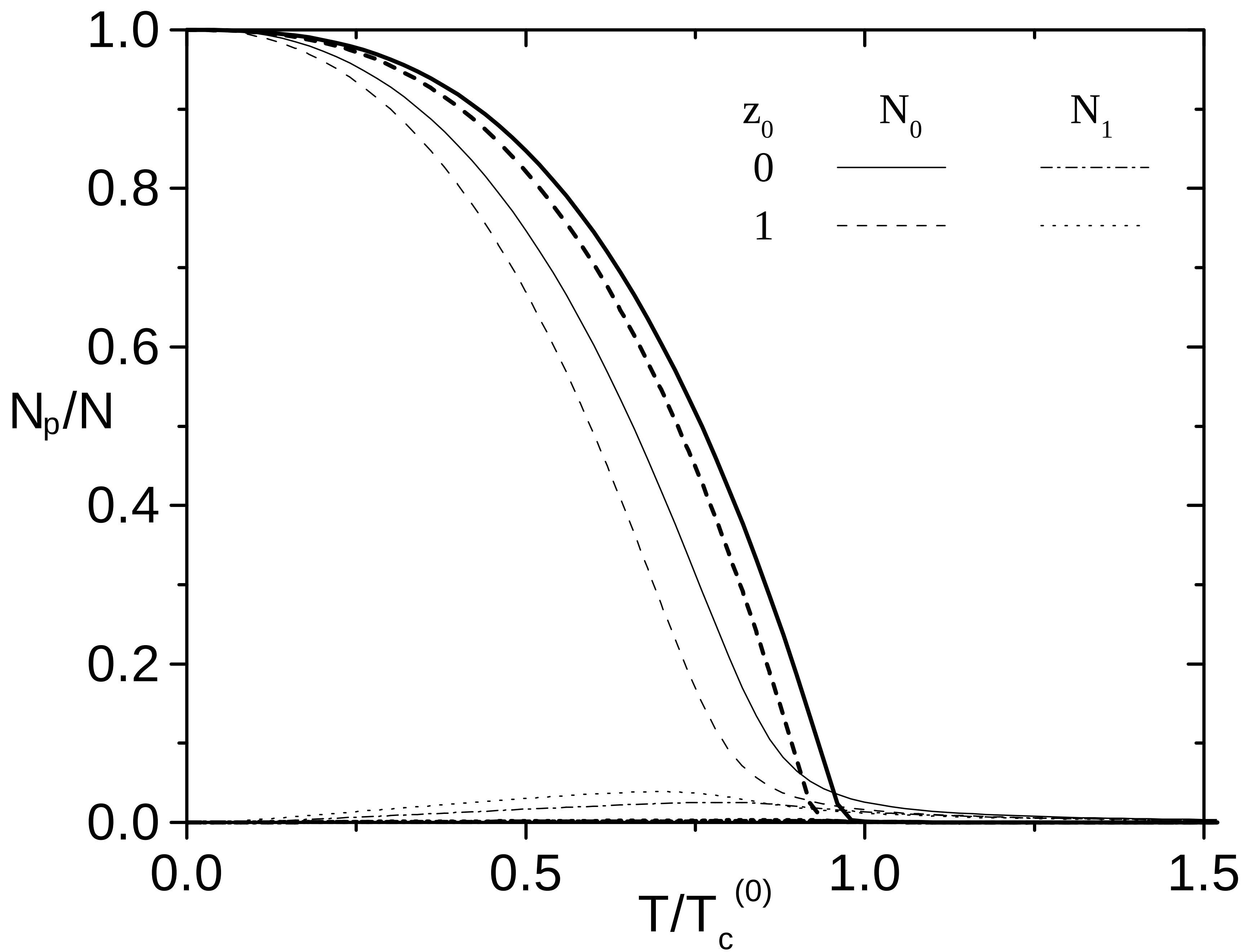
Figure 9.- Energies of the ground and first excited states (full and dotted lines, respectively) of interacting particles in an isotropic double well, as functions of the separation z_0, for various interaction strengths from s=0 up to s=10^{-2} (bottom to top) in steps Δs=10^{-3}. The dashed line indicates the barrier height V_0. Energies are given in units of $\hbar\omega$.



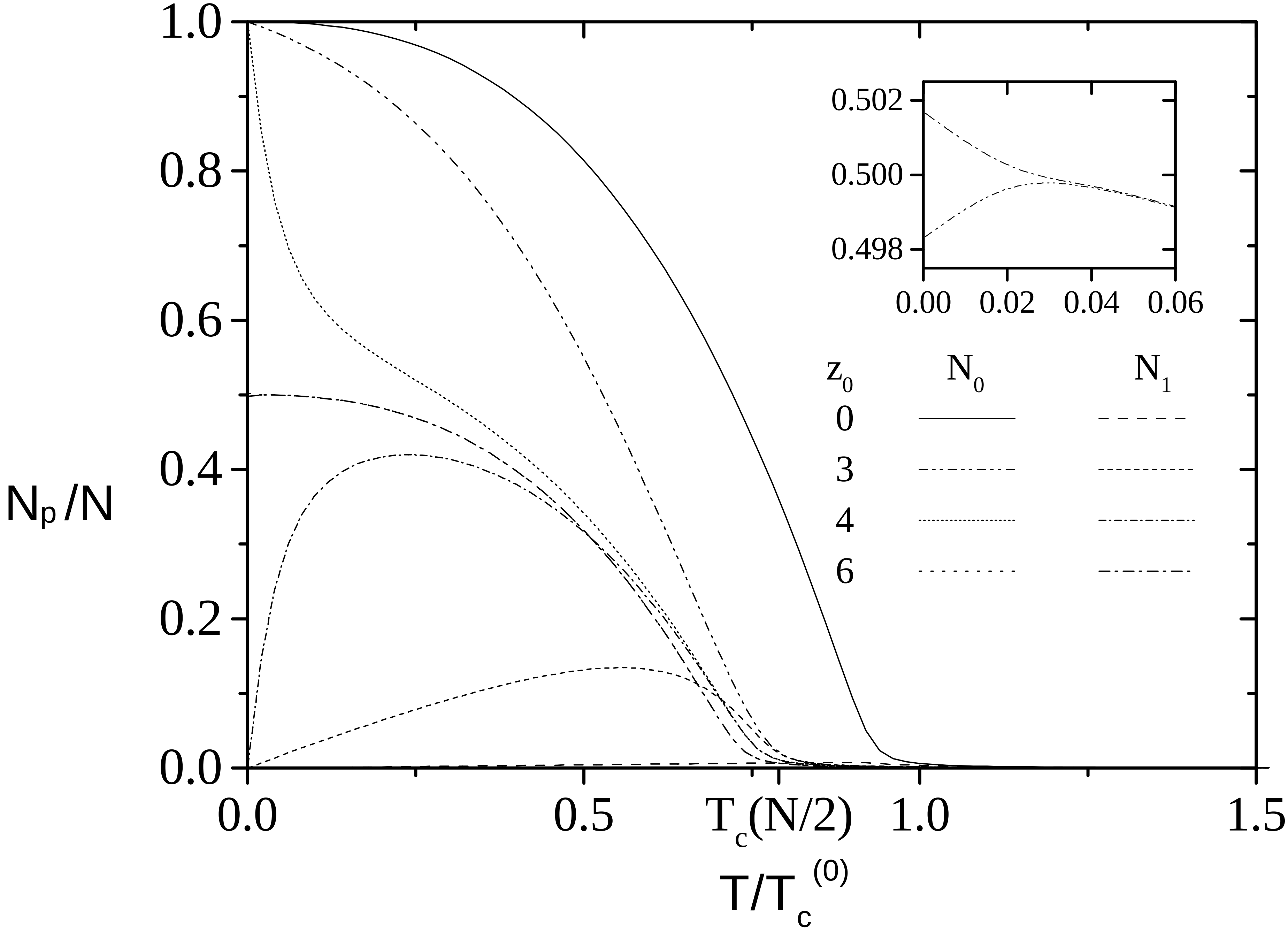
1.0
0.8
0.6
0.4
0.2
0.0
C_v/Nk_B
0
1
2
3
4
5
$T/T_c^{(0)}$
z_0
0
1
6



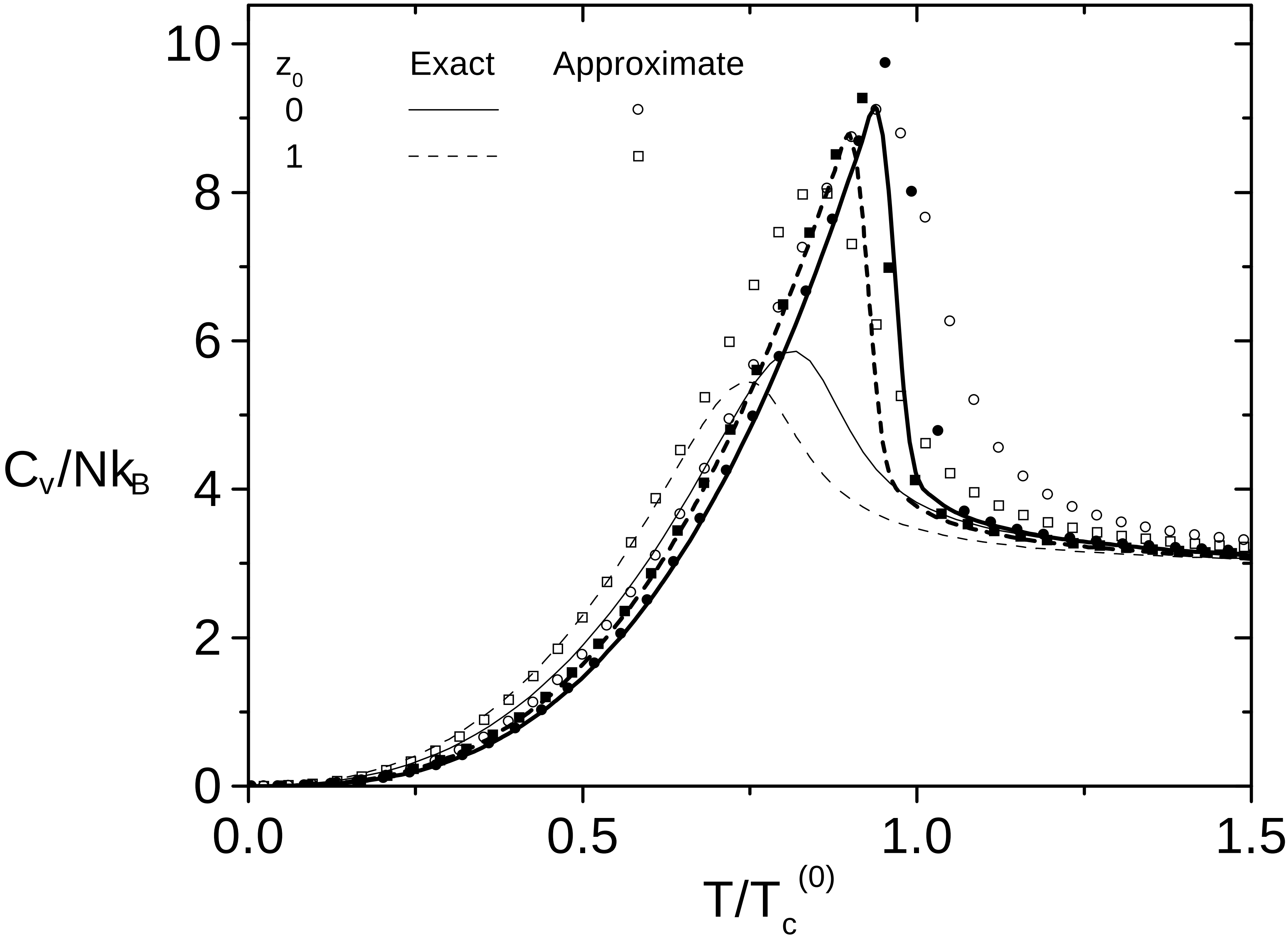
$|\psi|^2$
0.6
0.4
0.2
−6
−3
0
3
6
z/a



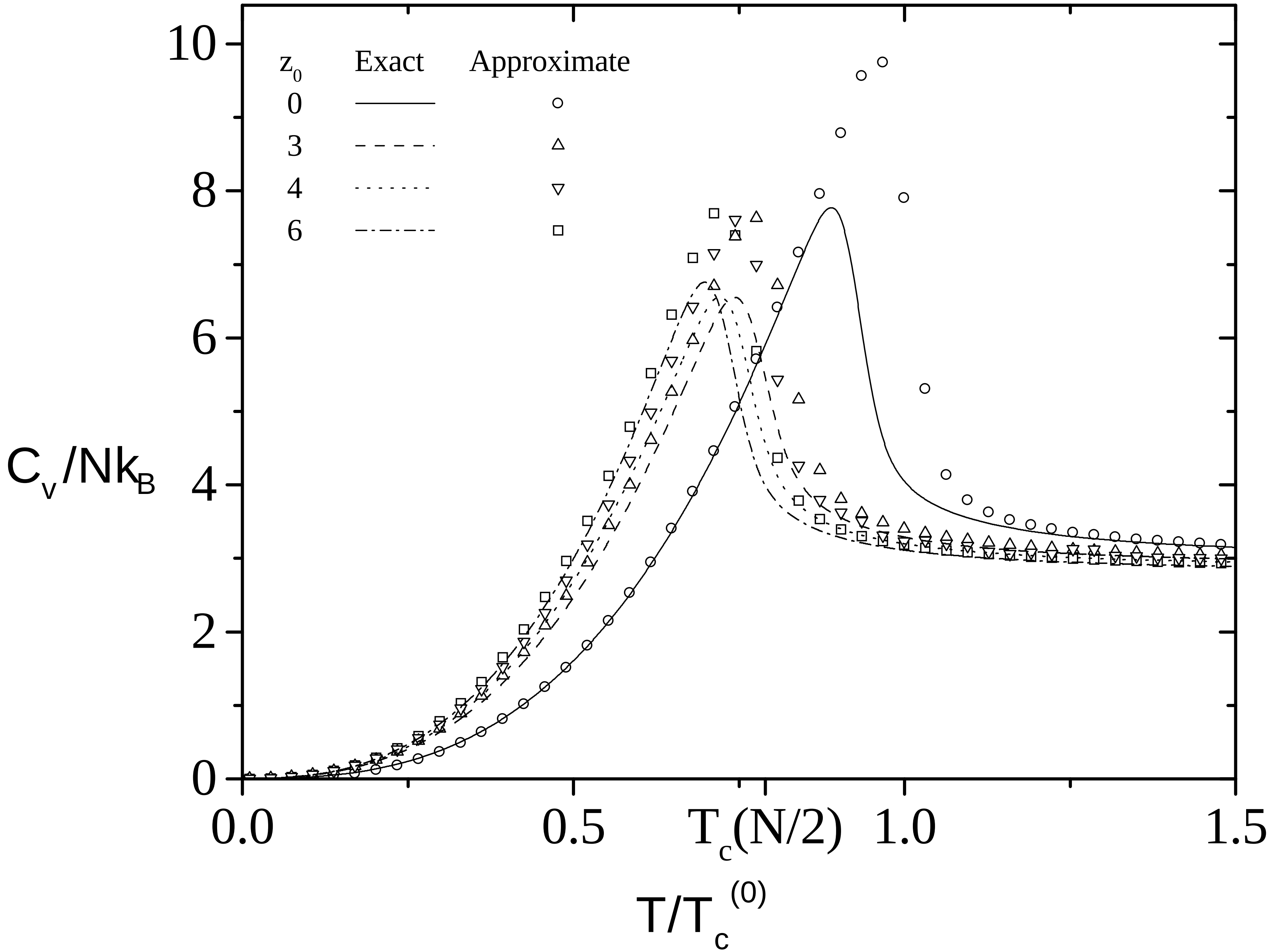
1.0
0.8
0.6
0.4
0.2
0.0
N_p/N
0.0
0.5
1.0
1.5
$T/T_c^{(0)}$
z_0
N_0
N_1
0
1



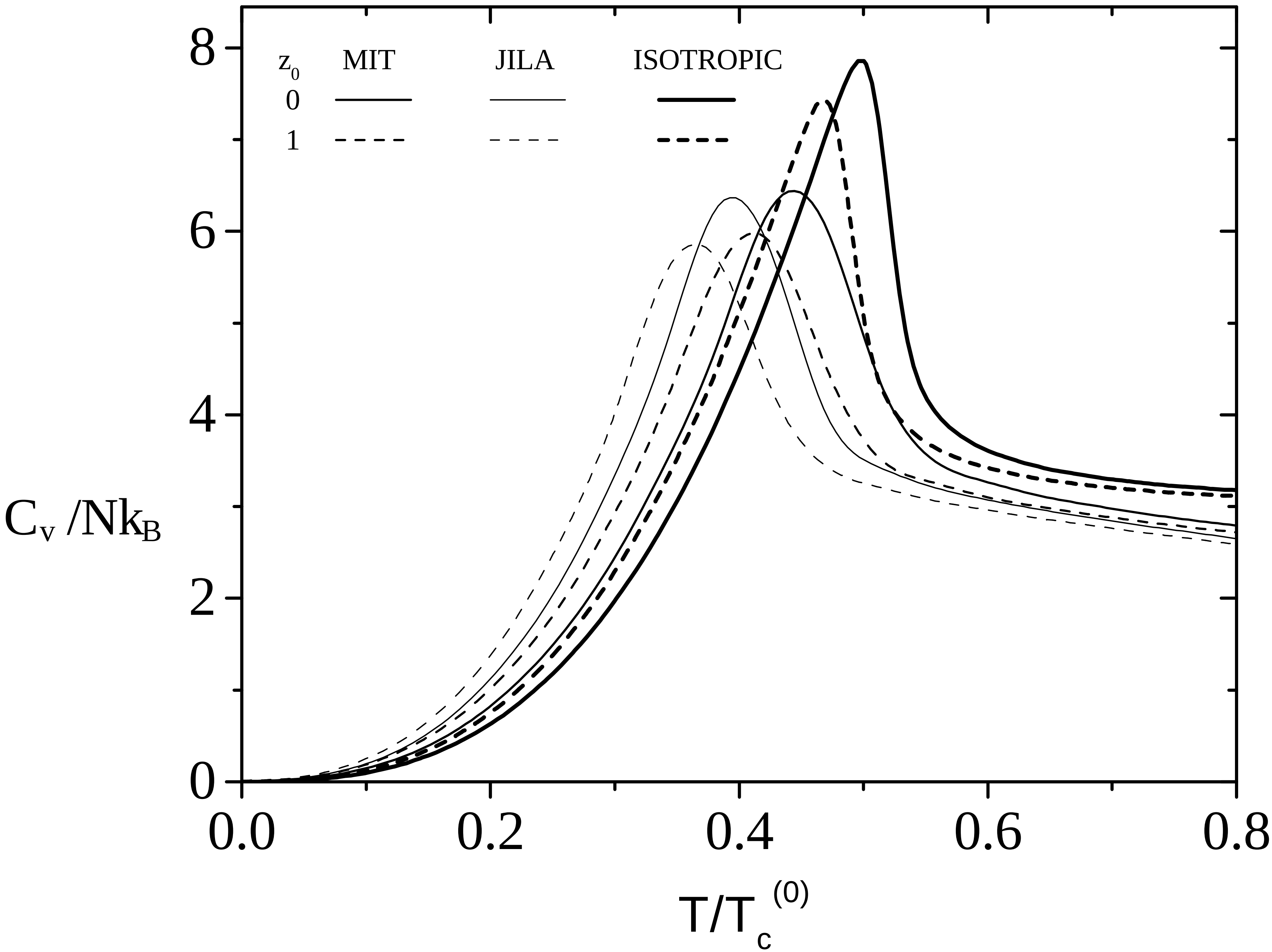
1.0
0.8
0.6
0.4
0.2
0.0
Np /N
0.0
0.5
Tc(N/2)
1.0
1.5
T/Tc(0)
0.502
0.500
0.498
0.00
0.02
0.04
0.06
z0
N0
N1
0
3
4
6



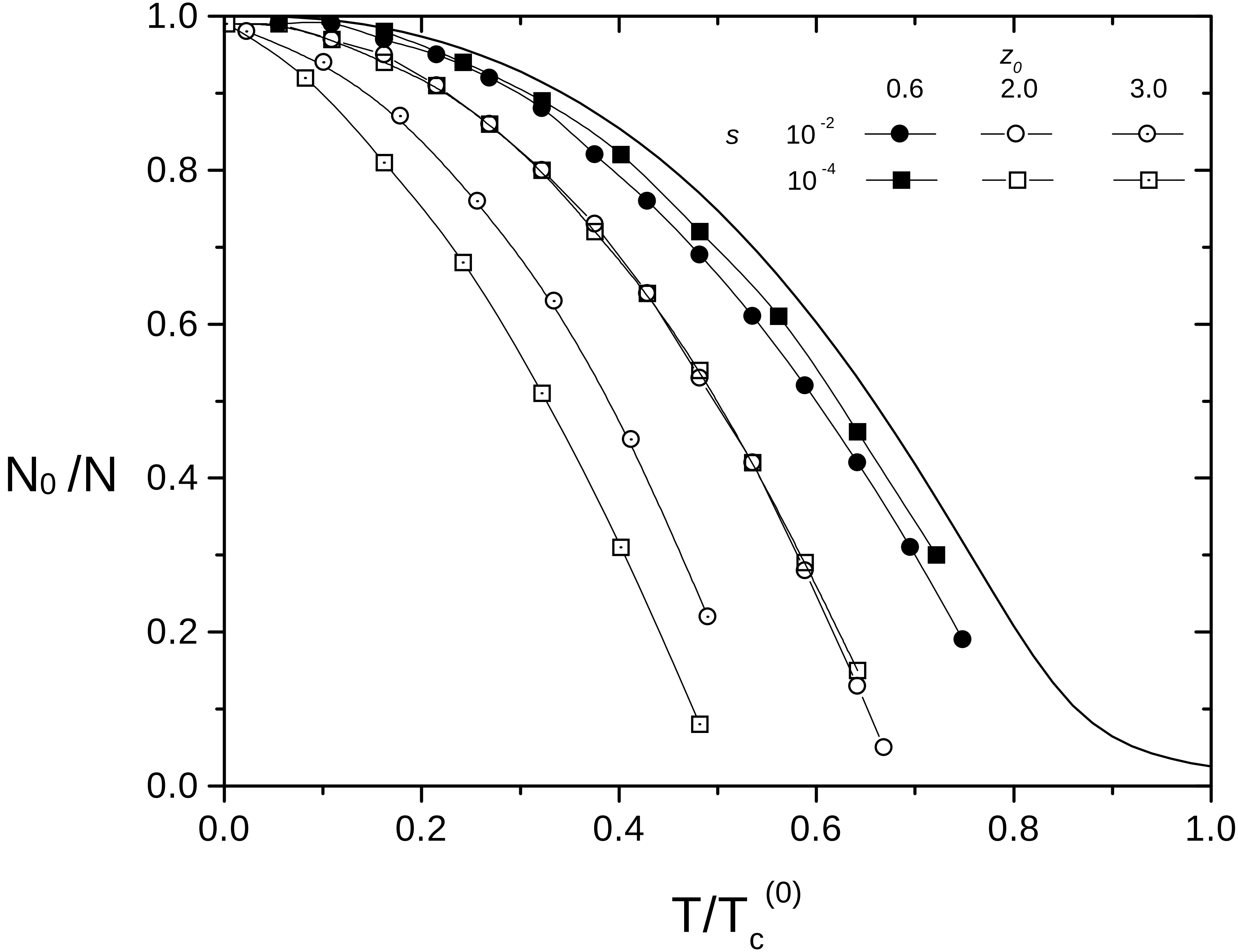
z_0
Exact
Approximate
0
1
C_v/Nk_B
$T/T_c^{(0)}$
0
2
4
6
8
10
0.0
0.5
1.0
1.5



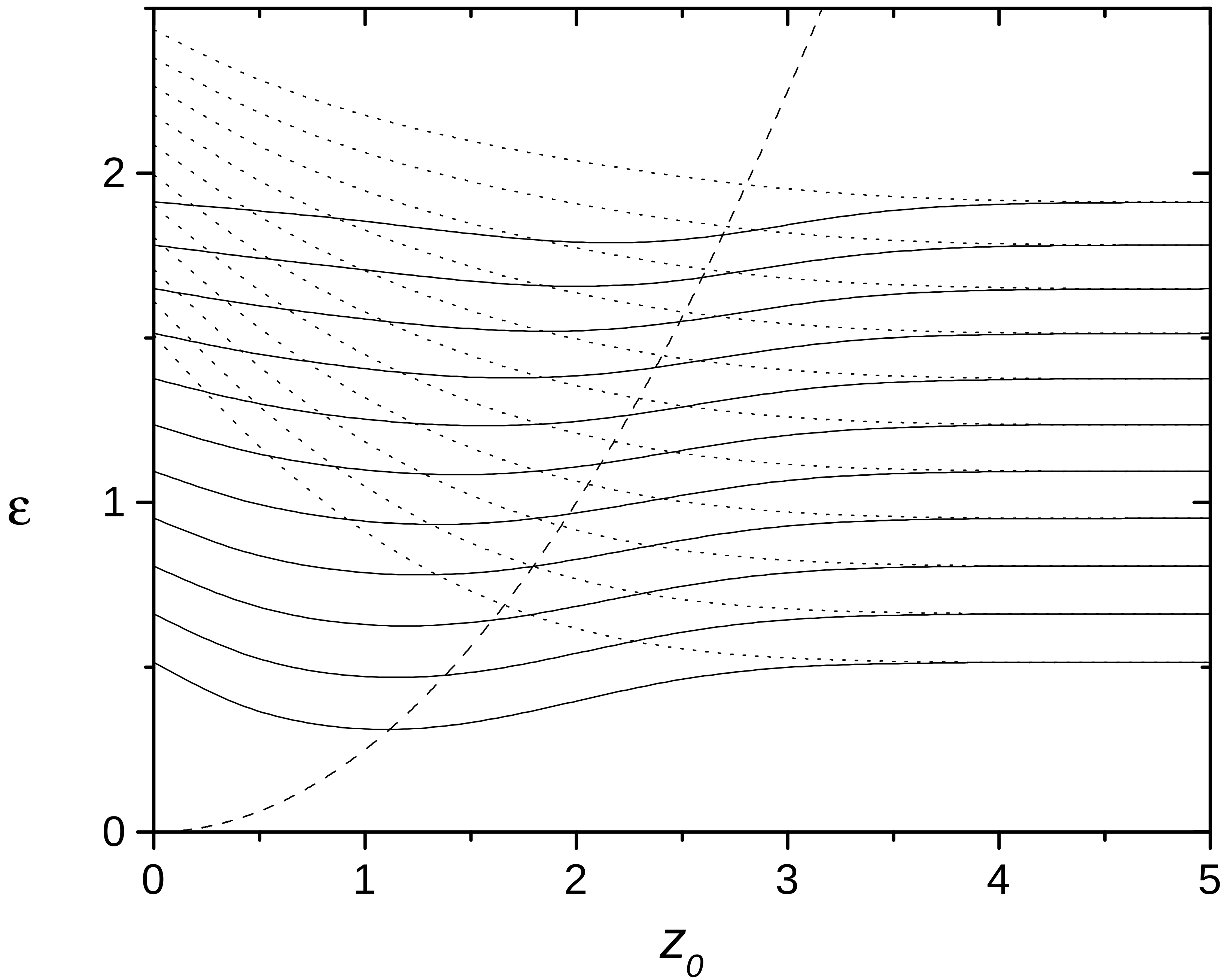
C_v/Nk_B
$T/T_c^{(0)}$
$T_c(N/2)$
z_0 Exact Approximate
0
3
4
6
0.0
0.5
1.0
1.5
0
2
4
6
8
10



8
6
4
2
0
C_v / Nk_B
0.0
0.2
0.4
0.6
0.8
$T/T_c^{(0)}$
z_0
MIT
JILA
ISOTROPIC
0
1



N0 /N
T/Tc(0)
z0
0.6
2.0
3.0
s
10-2
10-4
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0



2
1
0
ε
0
1
2
3
4
5
z_0